

Do four problems of your choice. Cite any other students or resources which you used.

Exercise 1.

- (a) Let f be a rational function and $\mathcal{A}$ the attracting basin of an attracting fixed point p of f .
Prove that the grand orbit $\text{GO}(f, p)$ is a discrete subset of $\mathcal{A}$ and that if p is not exceptional, then the set of accumulation points of $\text{GO}(f, p)$ is $J(f)$.
- (b) Suppose that $q \in \mathcal{A}$ and $q \notin \text{GO}(f, p)$. Prove that if p is geometrically attracting, then $\text{GO}(f, q)$ is a discrete subset of $\mathcal{A}$ and the set of accumulation points of $\text{GO}(f, q)$ is $J(f) \cup \text{GO}(f, p)$.
- (c) Suppose that $q \in \mathcal{A}$ and $q \notin \text{GO}(f, p)$. Now assume that p is superattracting of local degree n with Böttcher conjugacy $\phi : U \rightarrow \mathbb{D}_r$. The modulus of ϕ extends to $|\phi| : \mathcal{A} \rightarrow \mathbb{R}$. Show that

$$\overline{\text{GO}(f, q)} = J(f) \cup \text{GO}(f, p) \cup \{z \mid |\phi|(z) = |\phi|(q)^{n^k} \text{ for some } k\}.$$

Exercise 2 (Green's function).

- (a) Let $f(z) \in \mathbb{C}[z]$ be a monic polynomial and let $|\phi| : \widehat{\mathbb{C}} \setminus K(f) \rightarrow \mathbb{R}$ be the modulus of the polynomial Böttcher conjugacy $\phi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}_R \rightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}_r$, $r \geq 1$ extended to the entirety of the basin of attraction of infinity. The *Green's function* of $K(f)$ is $G(z) = \log |\phi(z)|$.
Prove that G satisfies the following properties:
- (i) G is harmonic on $\widehat{\mathbb{C}} \setminus K(f)$,
 - (ii) $G(z) \rightarrow 0$ as $z \mapsto J(f)$,
 - (iii) $G(z) - \log |z| \rightarrow 0$ as $z \rightarrow \infty$.
- (b) Prove that G is the unique function that satisfies the three properties in part (d) and hence only depends on the set $K(f)$.
- (c) What is the Green's function for the filled Julia set of the polynomial $f(z) = z^2 - 2$?
Hint: The inverse Böttcher map can be given explicitly.
- (d) Determine the external ray lamination associated to the filled Julia set of $f(z) = z^2 - 2$.

Exercise 3.

- (a) Prove that the Julia set of the rational map $f(z) = (2/z - 1)^2$ is the entire Riemann sphere.
- (b) There is another proof that repelling fixed points are dense in the Julia set due to Fatou:
Suppose that $z_0 \in J(f)$ is not a critical value or fixed point. Let $g_1, g_2 : U \rightarrow \widehat{\mathbb{C}}$ be two local branches of f^{-1} near z_0 . Consider the functions

$$\phi_n(z) = \frac{(f^n(z) - g_1(z))(z - g_2(z))}{(f^n(z) - g_2(z))(z - g_1(z))}.$$

When do these functions attain the three values $0, 1, \infty$?

Next prove that for some n , at least one of the three equations

$$f^n(z) = z, \quad f^n(z) = g_1(z), \quad f^n(z) = g_2(z)$$

has a solution inside U , and use this to complete the proof of the theorem.

- (c) If $z_0 \in J(f)$, then the definition of the Julia set tells us that there is an open set $U \ni z_0$ and a subsequence f^{n_i} which does not converge locally uniformly on U .

Prove that in fact a stronger statement is true: for any open set $U \ni z_0$, no subsequence f^{n_i} converges locally uniformly on U .

- (d) The definition of the Fatou set tells us that that on a Fatou component, every sequence of iterates of f has a subsequence which convergence locally uniformly. What are all such limits?

Exercise 4.

- (a) You may assume the results of Exercise 4 from Homework 2. Let $f_c(z) = z^2 + c$ be a quadratic polynomial with an attracting fixed point. (In the language of the referenced exercise, c lies in the main cardioid of the Mandelbrot set.) Prove that the only Fatou components of f_c are the immediate attracting basins of the fixed point and infinity.
- (b) Show that the only Fatou components of $f_{1/4}$ are the immediate parabolic basin of the fixed point $z = 1/2$ and the immediate attracting basin of infinity.
- (c) Prove that if f_c has an attracting 2-cycle, then there exist three bounded Fatou components U_1, U_0, U'_1 which map by the pattern $U_1 \leftrightarrow U_0 \leftarrow U'_1$, and that all remaining bounded Fatou components are connected components of $f_c^{-n}(U'_1)$.
- (d) In analogy with part (c), describe the way that f_c acts on its bounded Fatou components when it has a Siegel fixed point or a parabolic fixed point.
- (e) Prove that the transcendental map $f(z) = z + \sin(2\pi z)$ has a family of wandering domains U_n with $f(U_n) = U_{n+1} = U_n + 1$ and another family of wandering domains V_n with $f(V_n) = V_{n-1} = V_n - 1$.

Hint: Use Fractalstream to find the domains.

Exercise 5 (Parabolic fixed points).

- (a) In class I stated but did not prove the existence of parabolic linearizations:

Theorem 1. *Let p be a parabolic fixed point and suppose that the multiplier is equal to 1. For $\mathcal{A}$ a parabolic basin in some attracting direction, there exists a map $\alpha : \mathcal{A} \rightarrow \mathbb{C}$ which is injective on a petal $P \subset \mathcal{A}$ and which satisfies the identity $\alpha(f(z)) = \alpha(z) + 1$.*

In the course of proving that every attracting basin contains a critical point, I used Theorem 1 to prove

Theorem 2. *Let p be a parabolic fixed point and suppose that the multiplier is equal to 1. If P is an attracting petal of p , then P/f is isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$.*

By lifting to the universal cover, prove that Theorem 2 actually implies Theorem 1.

- (b) Show that $f(z) = z - 1/z$ has a parabolic fixed point at ∞ and that the two attracting parabolic basins are the upper and lower half planes. What is the Julia set of f ?
- (c) Prove that $f(z) = z - 1/z + 1$ has a parabolic fixed point at ∞ and that the Julia set is a Cantor set inside $\mathbb{R} \cup \{\infty\}$.

Exercise 6 (Local connectivity I).

- (a) Let $K \subset \mathbb{C}$ be a compact subset. Prove that K is locally connected if and only if ∂K is locally connected.
- (b) Consider the open subset U (in white) of the square depicted in Figure 1.

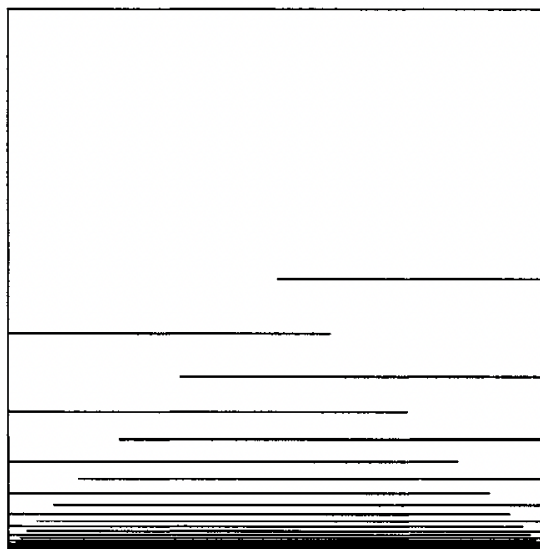


Figure 1: A non-locally connected set.

It is given by removing the segments

$$\bigcup_{k=1}^{\infty} ([1/(2k+1), 1] \times \{1/(2k+1)\}) \cup ([0, 1 - 1/(2k)] \times \{1/(2k)\})$$

from the interior of the unit square $[0, 1]^2$.

Prove that U is simply connected but that the boundary ∂U is not locally connected.

- (c) Endow U with its hyperbolic metric. Pick a basepoint $z_0 \in U \cap ([0, 1] \times (1/2, 1])$. Let $V_n \subset S^1$ be the set of unit vectors v at z_0 for which the geodesic ray in the direction v eventually crosses the segment $[0, 1] \times \{1/n\}$.

Prove that $V_n \supset \overline{V}_{n+1}$ and that $\bigcap_n V_n$ contains a vector $\hat{v}$.

- (d) Prove that every point in $[0, 1] \times \{0\}$ is an accumulation point of the geodesic ray $p(t)$ starting at z_0 in the direction $\hat{v}$.

Show that there exists an isomorphism $f : U \rightarrow U$ for which $f(p(t)) = p(t+1)$, and prove that every orbit in U of f accumulates on the boundary segment $[0, 1] \times \{0\}$. This shows that the *complément* in the “escape” case of the classification of dynamics on hyperbolic Riemann surfaces is not automatic.

- (e) Consider a hedgehog

$$H = \bigcup_{p/q \in \mathbb{Q} \cap [0, 1]} \{re^{2\pi i p/q} \mid 0 \leq r \leq 1/q\}.$$

Prove that H is compact and locally connected. Prove that the set of external rays for H landing at the center point is a Cantor set. Describe informally or with a picture what the external ray lamination looks like for H .

Exercise 7 (Local connectivity II).

- (a) Let $\psi : \mathbb{D} \rightarrow U \subset \widehat{\mathbb{C}}$ be an isomorphism and $p : [0, \infty) \rightarrow U$ be a continuous path which lands at a point of ∂U . Prove that $\psi^{-1} \circ p$ lands at a point of $\partial \mathbb{D}$.

Hint: Let $e^{2\pi i\theta} \in \partial \mathbb{D}$ be an accumulation point of the path $\psi^{-1} \circ p$. As in the proof of the Carathéodory-Torhorst theorem, consider a sequence of rectangles bounded by cross-cuts in the band $B \cong \mathbb{D}$ converging towards $e^{2\pi i\theta}$. Show that the path $\psi^{-1} \circ p$ eventually enters every neighborhood of $e^{2\pi i\theta}$ bounded by a cross-cut.

- (b) Prove the following theorem:

Theorem 3 (Lindelöf). *Let $\psi : \mathbb{D} \rightarrow U \subset \widehat{\mathbb{C}}$ be an isomorphism. Assume that there is a path $p : [0, \infty) \rightarrow U$ which lands at a point $e^{2\pi i\theta} \in \partial \mathbb{D}$ and whose image $\psi \circ p$ lands at a point $z_0 \in \partial U$. Then the ray $R_\theta = \psi(\{re^{2\pi i\theta} \mid r < 1\})$ also lands at z_0 .*

Hint: Consider a shrinking sequence of cross-cut neighborhoods through which $\psi \circ p$ enters as it converges towards z_0 . Prove that R_θ also enters these neighborhoods as well, and by estimating its hyperbolic distances, show that it also lands.

- (c) Consider the double comb from class. It is depicted in Figure 2 and given explicitly by the union

$$C = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}) \cup \bigcup_{n \neq 0} \{1/n\} \times [0, 1].$$

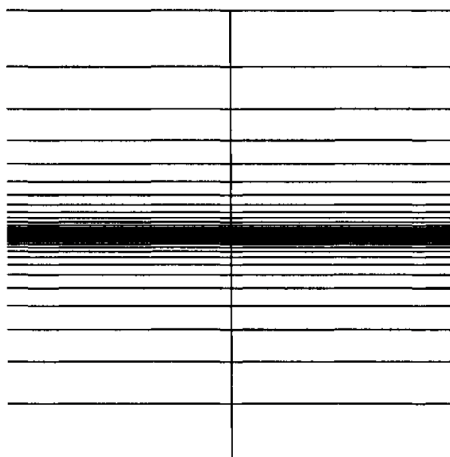


Figure 2: The double comb.

Prove that every external ray for the Riemann map of the complement $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus C$ lands at a point of C . Which points of C are not landing points of external rays?

Exercise 8 (Rotation number). Let $f : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow S^1$ be a homeomorphism of the circle and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a periodic lift of f . The *rotation number* of f is defined to be

$$R(f) = \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n} \in \mathbb{R}/\mathbb{Z}. \quad (1)$$

- (a) Prove that if the limit in (1) exists for some $x \in \mathbb{R}$, then it exists for any other $y \in \mathbb{R}$ and any other lift of f and is independent of these choices.
- (b) Suppose that f has a periodic point. Prove that the limit in (1) exists and is rational.
- (c) Suppose that f has no periodic points. Show that for all n , there exists an integer $p(n)$ such that $p(n) - 1 < F^n(x) < p(n)$ for all $x \in \mathbb{R}$. Deduce that

$$m(p(n) - 1) < F^{mn}(x) - x < m \cdot p(n).$$

Using this estimate, show that $p(n)/n$ is a Cauchy sequence. From there, prove the convergence of the limit in (1).

- (d) Prove that if $g : S^1 \rightarrow S^1$ is another homeomorphism, then $R(f) = R(g \circ f \circ g^{-1})$.
- (e) Prove that if $R(f)$ is rational, then f has a periodic point.
- (f) Prove that R is a continuous function of f in the C^0 topology. In addition, show that if F is a lift of f , then the rotation $R(F_\alpha)$ of the maps

$$F_\alpha(t) = F(t) + \alpha$$

is an increasing function of $\alpha \in \mathbb{R}$.

Exercise 9 (Herman rings). Assume the results of Exercise c. The goal of this exercise will be to show the existence of Herman rings for certain Blaschke products.

- (a) Prove that no polynomial can have a Herman ring.
- (b) Let $\kappa \geq 2$. A number $\theta \in \mathbb{R}$ is κ -*Diophantine* if there exists an $\varepsilon > 0$ such that

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^\kappa}$$

for all rational p/q . We say that θ is *Diophantine* if it is κ -Diophantine for some κ .

Prove that the set of Diophantine numbers has full measure.

- (c) We will use the following theorem:

Theorem 4 (Arnold). *Let θ be Diophantine and $\sigma > 1$. Then there exists an $\varepsilon > 0$ such that for all homeomorphisms $f : S^1 \rightarrow S^1$ with rotation number $R(f) = \theta$ which extend to holomorphic and injective maps on the annulus $\{\sigma^{-1} < |z| < \sigma\}$ and admit the bound $|f(z) - e^{2\pi i \theta} z| < \varepsilon$, f is holomorphically conjugate to the rotation $z \mapsto e^{2\pi i \theta} z$ on the annulus $\{\sigma^{-1/2} < |z| < \sigma^{1/2}\}$.*

Let

$$f_t(z) = e^{2\pi i t} \frac{z^2(z-a)}{(1-az)}.$$

Now choose $a > 1$ and $t \in \mathbb{R}$ so that Arnold's theorem applies to f_t . Deduce that f_t has a Herman ring. Hint: First choose a and then, using part (f) of Exercise , adjust t in order to get the right rotation number.

(d) Prove that the Herman ring of the map f_c surrounds a superattracting basin.

Exercise 10. This exercise proves the theorem of F. and M. Riesz which was cited in class.

Theorem 5 (F. and M. Riesz). *Suppose that $f : \mathbb{D} \rightarrow \mathbb{C}$ is bounded, nonconstant, and holomorphic. Then for every $c_0 \in \mathbb{C}$, the set*

$$\{\theta \in \mathbb{R}/2\pi\mathbb{Z} \mid \lim_{r \rightarrow 1^-} f(re^{i\theta}) = c_0\}$$

has Lebesgue measure zero.

(a) Consider the function

$$A(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

Prove that A is increasing.

(b) Suppose that the set of all vanishing radial limits

$$E = \{\theta \in \mathbb{R}/2\pi\mathbb{Z} \mid \lim_{r \rightarrow 1^-} f(re^{i\theta}) = 0\}$$

has positive measure and that $f : \mathbb{D} \rightarrow \mathbb{D}$. Let

$$E(\varepsilon, \delta) = \{\theta \in \mathbb{R}/2\pi\mathbb{Z} \mid 1 - \delta < r \implies |f(re^{i\theta})| < \varepsilon\}.$$

Write a set-theoretic expression that relates the sets $E(\varepsilon, \delta)$ to the set E . From this, prove that for any $\varepsilon > 0$, $E(\varepsilon, \delta)$ will have positive measure for all sufficiently small δ .

(c) Prove that for every $\varepsilon > 0$,

$$2\pi A(r) < \log(\varepsilon) \frac{|E|}{2}$$

for all r sufficiently close to 1. Deduce Theorem 5.

Exercise 11. Without using a pre-existing fractal plotting software, plot pictures of three filled Julia sets for three nontrivial (not z^n !) polynomials. Describe how you carried this out. Draw equipotential lines and external rays for the Böttcher conjugacy at infinity.

Exercise 12. Pick any exercise which you did not already complete from a previous problem set.